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VIBRATIONS OF RECTANGULAR MEMBRANES AND PLATES WITH RECTANGULAR HOLES WITH FIXED BOUNDARIES

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1. Introduction

Vibrating, rectangular plates with different combinations of boundary conditions and circular and rectangular cutouts have been the subject of several investigations in the case where the edges of the cutouts are free [1–3]. Vibrational characteristics of these structural elements are of interest in many situations of engineering (aeronautical, civil, mechanical, etc.).

On the other hand, if the boundary of the hole is not free, e.g., simply supported or clamped, the problem of transverse vibrations is considerably more complicated from the point of view of an analytical solution and, in general, one must make use of a numerical approach like the powerful finite element method. The situation is similar when solving the problem of transverse vibrations of a membrane, governed by the Helmholtz equation, when it is fixed at both boundaries.

In the case of a membrane of regular polygonal shape with a concentric circular boundary, solutions have been attained using the conformal mapping approach [4] or by constructing co-ordinate functions which are null at the outer and inner boundaries [5]. For instance in the case of a square domain with a fixed concentric circular perforation Laura *et al*. [5] determined the fundamental frequency coefficient by approximating the fundamental mode of vibration by means of

$$\psi \simeq \psi_a = (x^2 - a_p^2)/(y^2 - a_p^2)(\sqrt{x^2 + y^2} - R_0)\,[A_0 + A_1(x^2 + y^2)],$$

where a_p is the apothem and R_0 is the radius of the inner boundary.

A similar approach is employed in the present study in the case of a square cutout using a hyperelliptic† representation of a "quasi-square" [6] as it will be shown in the next section. Finally, the finite element method is employed to obtain the fundamental frequency of transverse vibration of square plates with square cutouts for the following combinations of boundary combinations. Outer boundary SS, SS, C, and C. Inner boundary SS, C, SS and C.

2. Vibrating rectangular membrane with a rectangular cutout of fixed edges

Consider the functional relation

$$\frac{(\bar{x} - \bar{x}_c)^n}{a_1^n} + \frac{(\bar{y} - \bar{y}_c)^n}{b_1^n} = 1 \qquad (1)$$

When $N = 2$ one has the classical equation of an ellipse with its center at $(\bar{x}_c, \bar{y}_c)$. For n sufficiently large and being an even number (say $n = 50$) one has the case of a hyperelliptic domain which is quite close to a rectangle (or a square if $a_1 = b_1$), see Figure 1.

† Also defined as "super elliptical" domains [6].

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Expression (1) can be expressed in the convenient form

$$(x - x_c)^n + (\eta_1/\eta)^n\ (y - y_c)^n = \alpha^n, \tag{2}$$

where

$$\eta = a/b, \qquad \eta_1 = a_1/b_1, \qquad \alpha = a_1/a, \qquad \beta = b_1/b = (a_1/\eta_1)/(a/\eta) = (a_1/a)(\eta/\eta_1) = \alpha(\eta/\eta_1),$$
$$x = \bar{x}/a, \qquad y = \bar{y}/b.$$

Following reference [5] one can now construct the following two-term approximation for the fundamental mode shape

$$W \cong W_\alpha = C_1\varphi_1(x, y) + c_2\varphi_2(x, y), \tag{3}$$

where

$$\varphi_1(x, y) = (x - x^P)(y - y^P)g(x, y),$$
$$\varphi_2(x, y) = (x - x^{P+1})(y - y^{P+1})g(x, y),$$

$g(x, y) = [(x - x_c)^n + (\eta_1/\eta)^n(y - y_c)^n]^{1/2} - \alpha$, and p is Rayleigh's optimization parameter [7].

It is quite easy to show that equation (3) satisfies Dirichlet's boundary condition at the outer and inner contours.

Use will be made now of the classical Rayleigh-Ritz method which requires minimization of the functional

$$J(W) = \iint_{\bar{D}} (W_{\bar{x}}^2 + W_{\bar{y}}^2)\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y} - \rho\omega^2/S \iint_{\bar{D}} W^2\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}, \tag{4}$$

which in terms of the dimensionless variables x and y becomes

$$\eta J(W) = \iint_D (W_{\bar{x}}^2 + \eta^2 W_{\bar{y}}^2)\,\mathrm{d}x\,\mathrm{d}y - \lambda^2 \iint_D W^2\,\mathrm{d}x\,\mathrm{d}y, \tag{5}$$

where $\lambda^2 = \rho a^2\omega^2/S$.

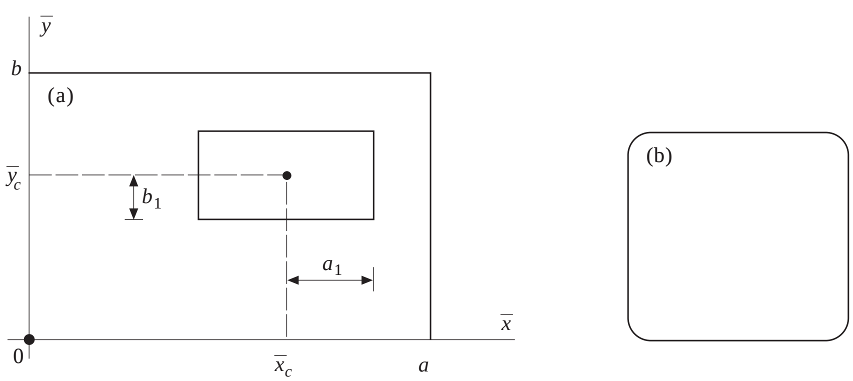

Figure 1. Geometry of the vibrating system under study. (a) membrane or plate executing transverse vibrations. (b) Representation of the inner boundary: hyperelliptic domain obtained for $a_1 = b_1$ ($n = 50$).

Substituting equation (3) in equation (5) results in

$$\frac{\eta}{2}\frac{\partial J}{\partial C_i} = \sum_{j=1}^{2}\left[\iint_D (\varphi_{jx}\varphi_{ix} + \eta^2\varphi_{jy}\varphi_{iy})\,\mathrm{d}x\,\mathrm{d}y - \lambda^2\iint_D \varphi_j\varphi_i\,\mathrm{d}x\,\mathrm{d}y\right]C_j = 0 \quad \text{for i} = 1, 2. \tag{6}$$

The non-triviality condition yields a 2×2 determinantal equation whose lowest root constitutes the fundamental frequency coefficient λ_1. Since

$$\lambda_1 = \lambda_1(p) \tag{7}$$

and λ_1 is an upper bound with respect to the exact value, by minimizing it with respect to p one obtains an optimized value of λ_1 [7].

3. FINITE ELEMENT ALGORITHMIC PROCEDURE

The present study makes use of the conforming rectangular element of 16 degrees of freedom by Bogner, Fox and Schmit [8] which yields very good accuracy, as can be ascertained from the fact that in the case where the four edges of a square plate are simply supported one obtains $\Omega_1 = \sqrt{\rho h/D}\,\omega_1 a^2 = 19{\cdot}739213$ and $\Omega_5 = \Omega_6 = \sqrt{\rho h/D}\,\omega_{5,6}\,a^2 = 98{\cdot}6988$ when one quarter of the square plate is subdivided into a net of 10×10 elements.

When the four edges of the square plate are clamped the same discrete model yields $\Omega_1 = 35{\cdot}98536$; $\Omega_6 = 132{\cdot}21267$. Both sets of eigenvalues practically coincide with the results quoted by Leissa in his classical treatise [1].

In order to obtain a high accuracy in the case of the structural system under study a convenient number of finite elements was chosen for each particular configuration, e.g., in the case of a square plate with a concentric, fixed cutout of half of the dimensions of the plate, 108 elements were used for one quarter of the structural element.

4. NUMERICAL RESULTS AND CONCLUSIONS

Table 1 depicts values of the fundamental frequency coefficient λ_1 as a function of $2a_1/a = 2\alpha$ and $a/b = \eta$ in the case of concentric configurations. For all the situations, $\eta = \eta_1$, that is $a/b = a_1/b_1$. The effect of dynamic stiffening is clearly observed. Table 2 shows the variation of λ_1 as the center of a square hole with fixed edges displaces along the x-axis in the case of a square membrane ($2\alpha = 0{\cdot}20$) while Table 3 depicts the variation of λ_1 as the center of the square cutout displaces along the diagonal of the square membrane. Table 4 depicts the variation of the fundamental frequency coefficient of square plates with different arrangements of edge conditions for the outer and inner boundaries, obtained by means of the finite element method.

In view of the equivalence of vibration problems and elastic stability phenomena in the case of polygonal plates with simply supported edges when the plate is subjected to a hydrostatic state of in-plane stress [1], one concludes that the first row of values of Table 4 is also the critical buckling parameter $N_{cr}\,a^2/D$. It is interesting to point out that if one calculates the square root of the first row one obtains the values 8·61, 9·49, 10·54, and 13·46, which are the fundamental frequency coefficients of square membranes with concentric, perfect square cutouts. The values of the first row of Table 1 of the present study are 7·424, 8·403, 9·464, and 11·738, which are lower than the corresponding previous eigenvalues in view of the fact that these eigenvalues correspond to square membranes with square cutouts with rounded corners.

TABLE 1

Values of λ_1 in the case of a rectangular membrane with a concentric rectangular cut-out with rounded corners.

	η			
2α	1	1·25	1·50	2
1/6	7·424	8·403	9·464	11·738
1/4	8·192	9·273	10·443	12·953
1/3	9·149	10·356	11·663	14·466
1/2	12·039	13·627	15·346	19·035

TABLE 2

Values of λ_1 in the case of a square membrane with a square cutout when the hole center is displaced along the x-axis ($2\alpha = 0{\cdot}20$).

(x_c, y_c)			
(0·5, 0·5)	(0·6, 0·5)	(0·7, 0·5)	(0·8, 0·5)
7·720	7·336	6·498	5·640

TABLE 3

Values of λ_1 in the case of a square membrane with a square cutout when the center of the hole is displaced along a diagonal of the square ($2\alpha = 0{\cdot}20$)

(x_c, y_c)		
(0·6, 0·6)	(0·7, 0·7)	(0·8, 0·8)
6·980	5·918	5·107

Admittedly, "infinite" stressses will arise in the case of plates with perfect rectangular cutouts. If the re-entrant corners are curved, one will have regions with high concentration of stresses and they will reduce the values of the frequency coefficients as it has been shown by Leissa and coworkers in fundamental studies [9].

TABLE 4

Fundamental frequency coefficients $\Omega_1 = (\rho h/D)\,\omega_1 a^2 2$ for square plates with concentric square cutouts considering different arrangements of boundary conditions and for different values of the parameter $2a_1/a$ (Poisson's ratio, $\mu = 0{\cdot}30$)

Boundary		$2a_1/a$			
Outer	Inner	1/6	1/4	1/3	1/2
SS	SS	74·17	90·14	111·16	181·26
SS	C	76·44	95·87	122·96	222·83
C	SS	111·02	134·54	165·31	269·34
C	C	114·52	143·22	183·07	329·58

It is interesting to point out that by decreasing the size of the internal boundary one obtains by means of the finite element method for $2a_1/a = 1/12$, $\Omega_1 = 61{\cdot}73$, and for $2a_1/a = 1/24$, $\Omega_1 = 56{\cdot}25$. When the inner boundary reduces to a point support $\Omega_1 = 52{\cdot}66$, which is in good agreement with the result quoted by Leissa [1]: $\Omega_1 = 52{\cdot}6$.

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